The Bloch Sphere

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Introduction

The Bloch sphere is a geometric representation of qubit states as points on the surface of a unit sphere.

Many operations on single qubits that are commonly used in quantum information processing can be neatly described within the Bloch sphere picture.

Definition of the Bloch sphere

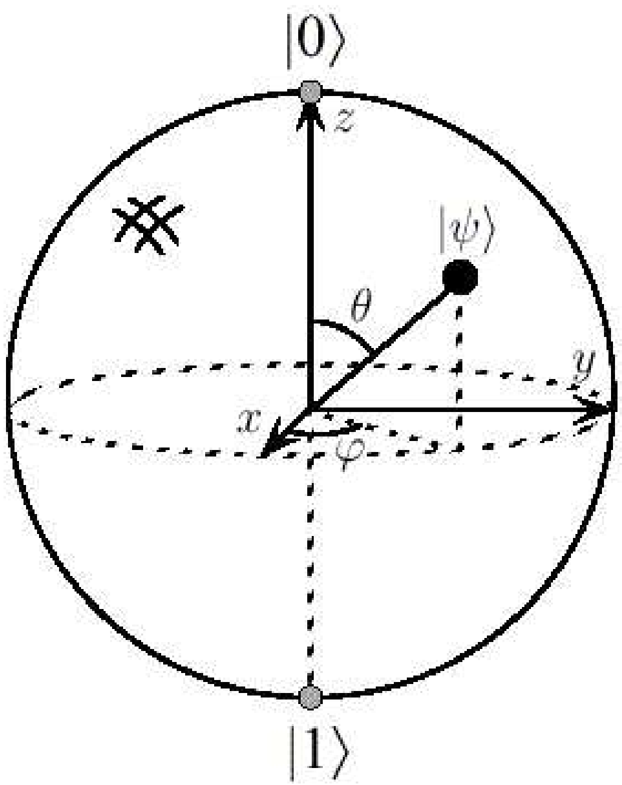
It turns out that an arbitrary single qubit state can be written:

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right)$$

where θ, ϕ and γ are real numbers. The numbers $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ define a point on a unit three-dimensional sphere. This is the *Bloch sphere*. Qubit states with arbitrary values of γ are all represented by the same point on the Bloch sphere because the factor of $e^{i\gamma}$ has *no observable effects*, and we can therefore effectively write:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

The Bloch Sphere



Derivation of the Bloch Sphere

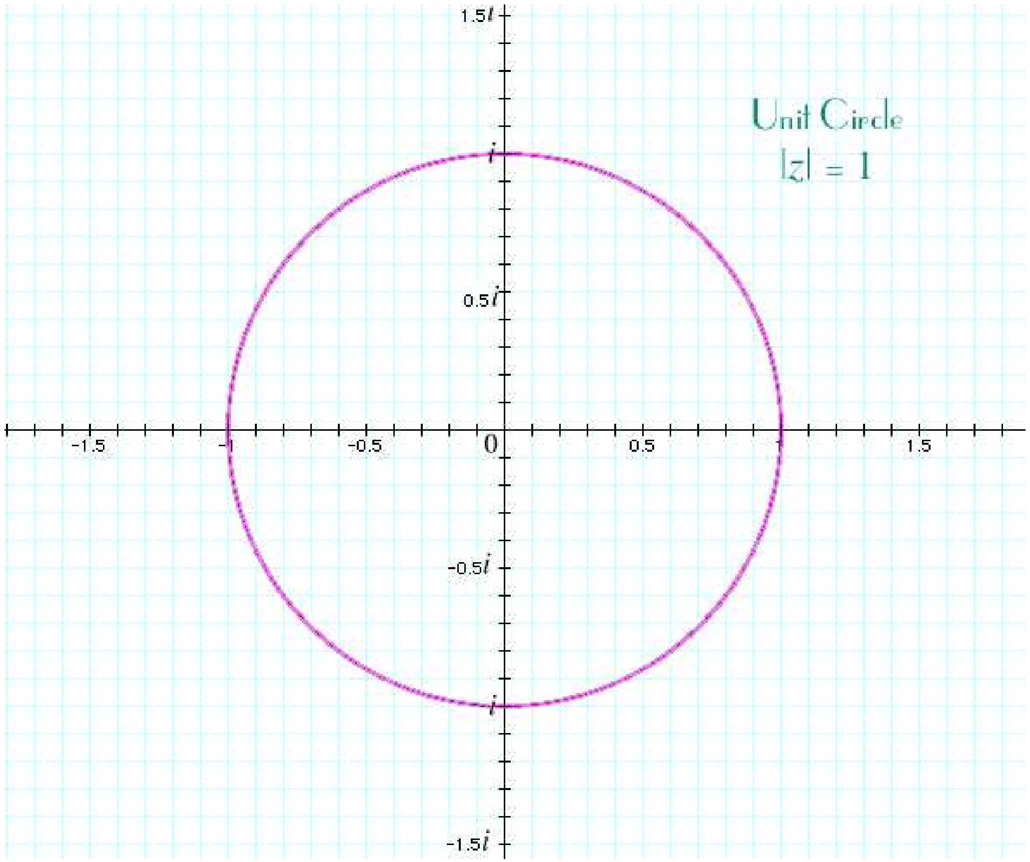
The Bloch Sphere is is a generalisation of the representation of a complex number z with $|z|^2 = 1$ as a point on the unit circle in the complex plane.

If $z = x + iy$, where x and y are real, then:

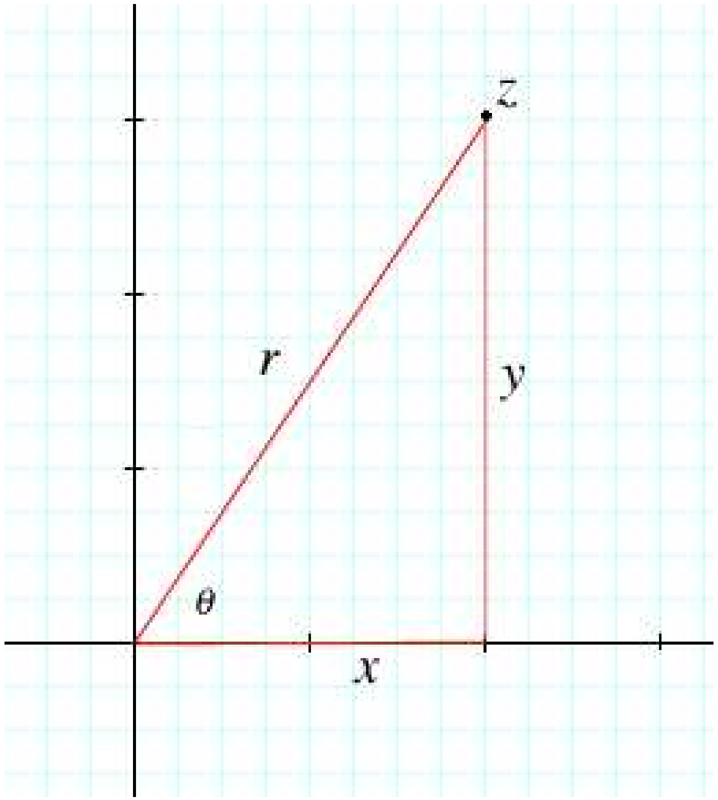
$$\begin{aligned} |z|^2 &= z^* z \\ &= (x - iy)(x + iy) \\ &= x^2 + y^2 \end{aligned}$$

and $x^2 + y^2 = 1$ is the equation of a circle of radius one, centered on the origin.

The Unit Circle



Polar Coordinates



Polar Coordinates

For arbitrary $z = x + iy$ we can write $x = r\cos\theta$, $y = r\sin\theta$, so

$$z = r(\cos\theta + i\sin\theta)$$

and using Euler's identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we have

$$z = re^{i\theta}$$

and the unit circle $(r = 1)$ can be written in the compact form:

$$z = e^{i\theta}$$

Notice that the constraint $|z|^2 = 1$ has left just one degree of freedom.

Qubit States

A general qubit state can be written

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with complex numbers α and β, and the normalization constraint $\langle\psi|\psi\rangle = 1$ requires that:

$$|\alpha|^2 + |\beta|^2 = 1$$

We can express the state in polar coordinates as:

$$|\psi\rangle = r_\alpha e^{i\phi_\alpha}|0\rangle + r_\beta e^{i\phi_\beta}|1\rangle$$

with four real parameters r_α, ϕ_α, r_β and ϕ_β.

Global Phase Invariance

However, the only measurable quantities are the probabilities $|\alpha|^2$ and $|\beta|^2$, so multiplying the state by an arbitrary factor $e^{i\gamma}$ (a global phase) has no observable consequences, because:

$$|e^{i\gamma}\alpha|^2 = (e^{i\gamma}\alpha)^*(e^{i\gamma}\alpha) = (e^{-i\gamma}\alpha^*)(e^{i\gamma}\alpha) = \alpha^*\alpha = |\alpha|^2$$

and similarly for $|\beta|^2$. So, we are free to multiply our state by $e^{-i\phi_\alpha}$, giving:

$$|\psi'\rangle = r_\alpha|0\rangle + r_\beta e^{i(\phi_\beta - \phi_\alpha)}|1\rangle = r_\alpha|0\rangle + r_\beta e^{i\phi}|1\rangle$$

which now has only three real parameters, r_α, r_β, and $\phi = \phi_\beta - \phi_\alpha$.

The Normalization Constraint

In addition we have the normalization constraint $\langle\psi'|\psi'\rangle = 1$

Switching back to cartesian representation for the coefficient of $|1\rangle$

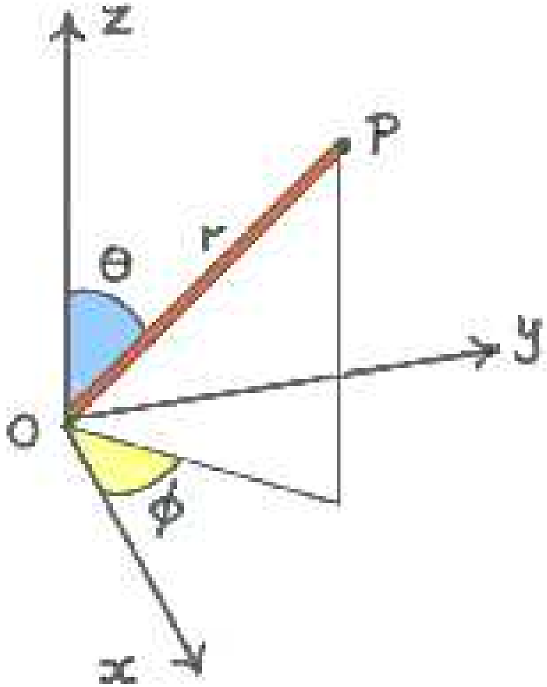
$$|\psi'\rangle = r_\alpha|0\rangle + r_\beta e^{i\phi}|1\rangle = r_\alpha|0\rangle + (x+iy)|1\rangle$$

and the normalization constraint is:

$$\begin{aligned} |r_\alpha|^2 + |x+iy|^2 &= {r_\alpha}^2 + (x+iy)^*(x+iy) \\ &= {r_\alpha}^2 + (x-iy)(x+iy) \\ &= {r_\alpha}^2 + x^2 + y^2 = 1 \end{aligned}$$

which is the equation of a unit sphere in real 3D space with cartesian coordinates (x, y, r_α)!

Spherical Polar Coordinates



Spherical Polar Coordinates

Cartesian coordinates are related to polar coordinates by:

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

so renaming r_α to z, and remembering that $r = 1$, we can write:

$$\begin{aligned} |\psi'\rangle &= z|0\rangle + (x + iy)|1\rangle \\ &= \cos\theta|0\rangle + \sin\theta(\cos\phi + i\sin\phi)|1\rangle \\ &= \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle \end{aligned}$$

Now we have just two parameters defining points on a unit sphere.

Half Angles

But this is still not the Bloch sphere. What about the half angles? Let

$$|\psi\rangle = \cos\theta'|0\rangle + e^{i\phi}\sin\theta'|1\rangle$$

and notice that $\theta' = 0 \Rightarrow |\psi\rangle = |0\rangle$ and $\theta' = \frac{\pi}{2} \Rightarrow |\psi\rangle = e^{i\phi}|1\rangle$

which suggests that $0 \leq \theta' \leq \frac{\pi}{2}$ may generate all points on the Bloch sphere.

Half Angles

Consider a state $|\psi'\rangle$ corresponding to the *opposite* point on the sphere, which has polar coordinates $(1, \pi - \theta', \phi + \pi)$

$$\begin{aligned}
|\psi'\rangle &= \cos(\pi - \theta')|0\rangle + e^{i(\phi+\pi)}\sin(\pi - \theta')|1\rangle \\
&= -\cos(\theta')|0\rangle + e^{i\phi}e^{i\pi}\sin(\theta')|1\rangle \\
&= -\cos(\theta')|0\rangle - e^{i\phi}\sin(\theta')|1\rangle \\
|\psi'\rangle &= -|\psi\rangle
\end{aligned}$$

So it is only necessary to consider the upper hemisphere $0 \leq \theta' \leq \frac{\pi}{2}$, as opposite points in the lower hemisphere differ only by a phase factor of -1 and so are equivalent in the Bloch sphere representation.

The Bloch Sphere

We can map points on the upper hemisphere onto points on a sphere by defining

$$\theta = 2\theta' \quad \Rightarrow \quad \theta' = \frac{\theta}{2}$$

and we now have

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ are the coordinates of points on the *Bloch sphere.*

The Bloch Sphere

- Notice that $\theta = 2\theta'$ is a one-to-one mapping except at $\theta' = \frac{\pi}{2}$, where all the points on the θ' 'equator' are mapped to the single point $\theta = \pi$, the 'south pole' on the Bloch sphere
- This is okay, since at the south pole $|\psi\rangle = e^{i\phi}|1\rangle$ and ϕ is a global phase with no significance. (Longitude is meaningless at a pole!)
- Rotations in a 2D complex vector space contain a double representation of rotations in 3D real space
- Formally, there is a 2 to 1 homomorphism of SU(2) on SO(3)
- Notice also that as we cross the θ'-equator going south, we effectively start going north again on the other side of the Bloch sphere, because opposite points are equivalent on the θ' sphere.

Properties of the Bloch Sphere

- Orthogonality of Opposite Points
- Rotations on the Bloch Sphere

Orthogonality of Opposite Points

Consider a general qubit state $|\psi\rangle$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

and $|\chi\rangle$ corresponding to the opposite point on the Bloch sphere

$$\begin{aligned} |\chi\rangle &= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle \\ &= \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle \end{aligned}$$

So

$$\langle\chi|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right)$$

Orthogonality of Opposite Points

$$\langle\chi|\psi\rangle = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right)$$

But $\cos(a+b) = \cos a \cos b - \sin a \sin b$, so

$$\langle\chi|\psi\rangle = \cos\frac{\pi}{2} = 0$$

and opposite points correspond to orthogonal qubit states.

Note that in the coordinate system we used in the derivation of the Bloch sphere, with $\theta' = \theta/2$, the two points are also orthogonal - 90° apart.

Rotations on the Bloch Sphere

The Pauli X, Y and Z matrices are so-called because when they are exponentiated, they give rise to the *rotation operators*, which rotate the Bloch vector $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ about the $\hat{x}$, $\hat{y}$ and $\hat{z}$ axes:

$$R_x(\theta) \equiv e^{-i\theta X/2}$$
$$R_y(\theta) \equiv e^{-i\theta Y/2}$$
$$R_z(\theta) \equiv e^{-i\theta Z/2}$$

In order to evaluate these exponentials, let's take a look at operator functions.

Operator Functions

If A is a normal operator ($A^\dagger A = AA^\dagger$) with spectral decomposition $A = \sum_a a|a\rangle\langle a|$ then we can define

$$f(A) \equiv \sum_a f(a)|a\rangle\langle a|$$

It is possible to use this approach to evaluate the exponentials of the Pauli matrices, but it turns out to be simpler to use the equivalent power series definition of an operator function. If $f(x)$ has a power series expansion $f(x) = \sum_{i=0}^{\infty} c_i x^n$ then we have

$$f(A) \equiv c_0 I + c_1 A + c_2 A^2 + c_3 A^3 + \cdots$$

Operator Exponential

For the case of the exponential function, we therefore have

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \frac{A^5}{5!} + \cdots$$

Now, consider $e^{i\theta A}$

$$e^{i\theta A} = I + i\theta A - \frac{(\theta A)^2}{2!} - i\frac{(\theta A)^3}{3!} + \frac{(\theta A)^4}{4!} + i\frac{(\theta A)^5}{5!} + \cdots$$

and in the special case that $A^2 = I$

$$e^{i\theta A} = I + i\theta A - \frac{\theta^2 I}{2!} - i\frac{\theta^3 A}{3!} + \frac{\theta^4 I}{4!} + i\frac{\theta^5 A}{5!} + \cdots$$

Operator Exponential

$$
\begin{aligned}
e^{i\theta A} &= I + i\theta A - \frac{\theta^2 I}{2!} - i\frac{\theta^3 A}{3!} + \frac{\theta^4 I}{4!} + i\frac{\theta^5 A}{5!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) I + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) A \\
e^{i\theta A} &= \cos(\theta) I + i\sin(\theta) A
\end{aligned}
$$

Now, the Pauli matrices have the property that $X^2 = Y^2 = Z^2 = I$, so we can use this equation to evaluate the rotation operators.

The Rotation Operators

$$
\begin{aligned}
R_x(\theta) &\equiv e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \\
R_y(\theta) &\equiv e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \\
R_z(\theta) &\equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}
\end{aligned}
$$

An Example Rotation

Consider:

$$R_x(\pi) = \begin{bmatrix} \cos\frac{\pi}{2} & -i\sin\frac{\pi}{2} \\ -i\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -iX$$

Which is equal to X up to the global phase of $-i$, so we see that the X operator is equivalent to a rotation of 180° about the X axis.

We also see that the rotation operators do not in general keep the coefficient of the $|0\rangle$ component of the qubit state real.

To compare rotated states to see if they correspond to the same point on the Bloch sphere, it is necessary to multiply each one by a phase to make the $|0\rangle$ component of its state real.

Another Example

Now consider

$$R_z(\alpha)|\psi\rangle = \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} e^{-i\alpha/2}\cos\frac{\theta}{2} \\ e^{i\alpha/2}e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

In order to make the coefficient of $|0\rangle$ real, we have to multiply this state by a phase $e^{i\alpha/2}$, giving

$$e^{i\alpha/2}\begin{bmatrix} e^{-i\alpha/2}\cos\frac{\theta}{2} \\ e^{i\alpha/2}e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\alpha}e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

so the net effect is to change ϕ to $\phi + \alpha$, as you would expect for a rotation around the $\hat{z}$ axis.

Rotation About an Arbitrary Axis

If $\hat{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions, then it can be shown that the operator $R_{\hat{n}}(\theta)$ rotates the Bloch vector by an angle θ about the $\hat{n}$ axis, where

$$R_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{n} \cdot \vec{\sigma}/2)$$

and $\vec{\sigma}$ denotes the three component vector (X, Y, Z) of Pauli matrices. Furthermore, it is not hard to show that $(\hat{n} \cdot \vec{\sigma})^2 = I$, and therefore we can use the special case operator exponential and write

$$\begin{aligned} R_{\hat{n}}(\theta) &= \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) \hat{n} \cdot \vec{\sigma} \\ &= \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) (n_x X + n_y Y + n_z Z) \end{aligned}$$

Arbitrary Unitary Operator

It can be shown that an arbitrary single qubit unitary operator can be written in the form

$$U = \exp(i\alpha)R_{\hat{n}}(\theta)$$

For some real numbers α and θ and a real three-dimensional unit vector $\hat{n}$. For example, consider $\alpha = \pi/2$, $\theta = \pi$, and $\hat{n} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

$$\begin{aligned} U &= \exp(i\pi/2)\left[\cos\left(\frac{\pi}{2}\right) I - i\sin\left(\frac{\pi}{2}\right)\frac{1}{\sqrt{2}}(X+Z)\right] \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

which is the Hadamard gate H.

Rotations and Phase

The double representation of rotations in 3D space has the interesting consequence that rotations of 360° do not restore the phase to its initial value, and a rotation through 720° is needed. For example:

$$\begin{aligned} R_z(0) &= I \\ R_z(2\pi) &= -I \\ R_z(4\pi) &= I \end{aligned} \tag{1}$$

For an isolated qubit this has no physical significance, but in relation to other qubits there is a difference

Orientation-Entanglement Relation

- e.g. a rotation of 360° of an electron about its 'spin' axis changes its state - a 720° rotation is needed to restore it
- What's more, this kind of phenomenon is not restricted to the world of quantum mechanics
- It affects the 'orientation-entanglement relation' of objects in everyday 3D real space
- And I have a demo!

Future Topics

- Proof that $R_{\hat{n}}(\theta)$ rotates the Bloch vector by an angle θ about the $\hat{n}$ axis
- Generalisation of the Bloch sphere to mixed states
- Generalizations to more qubits